AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 9: OCP Solution

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

A nonlinear control system with $n \times 1$ state $\boldsymbol{x}$, and $m \times 1$ control $\boldsymbol{u}$, is called driftless control-affine if it can be written in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{u},$$

where the $n \times m$ matrix $\boldsymbol{F}(\boldsymbol{x})$ has columns $\boldsymbol{f}_1(\boldsymbol{x}), \dots, \boldsymbol{f}_m(\boldsymbol{x})$. With fixed terminal time T, consider the following minimum energy OCP for such a control system:

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & \int_0^T \frac{1}{2}\|\boldsymbol{u}\|_2^2 \,\mathrm{d}t \\ \text{subject to} \quad & \dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{u}, \\ & \boldsymbol{x}(t=0) = \boldsymbol{x}_0 \text{ given}, \quad \boldsymbol{x}(t=T) = \boldsymbol{x}_T \text{ given}. \end{aligned} \tag{1}$$

All 5 questions below are for this problem (1).

Problem 1

A wheeled mobile robot with 3×1 state $\boldsymbol{x} := (x, y, \theta)^\top$, and 2×1 control $\boldsymbol{u} := (v, \omega)^\top$, has controlled dynamics

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{pmatrix}.$$

Is this system driftless control-affine?

(A) Yes. `Reason:` can re-write in desired form: $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}$.

(B) No.

(C) May or may not be depending on additional information.

Problem 2

In problem (1), the Hamiltonian H equals

(A) $\frac{1}{2}\boldsymbol{u}^\top\boldsymbol{u}$.

(B) $\boldsymbol{\lambda}^\top\boldsymbol{F}(\boldsymbol{x})\boldsymbol{u}$.

(C) $\frac{1}{2}\boldsymbol{u}^\top\boldsymbol{u}+\boldsymbol{\lambda}^\top\boldsymbol{F}(\boldsymbol{x})\boldsymbol{u}$. `Reason:` substitute $L=\frac{1}{2}\boldsymbol{u}^\top\boldsymbol{u}$, $\boldsymbol{f}=\boldsymbol{F}(\boldsymbol{x})\boldsymbol{u}$ in $H=L+\langle\boldsymbol{\lambda},\boldsymbol{f}\rangle$.

Problem 3

In problem (1), the transversality condition gives

(A) $\boldsymbol{\lambda}(T)=\boldsymbol{0}$.

(B) $H(T)=0$.

(C) $0+0=0$, therefore no new information. `Reason:` both final time T and final state $\boldsymbol{x}(T)$ are fixed.

Problem 4

In problem (1), the PMP $\boldsymbol{0}=\left.\frac{\partial H}{\partial \boldsymbol{u}}\right|_{\text{opt}}$ gives optimal control

(A) $\boldsymbol{u}^{\text{opt}}=\boldsymbol{F}(\boldsymbol{x}^{\text{opt}})\boldsymbol{\lambda}^{\text{opt}}$.

(B) $\boldsymbol{u}^{\text{opt}}=\left(\boldsymbol{F}(\boldsymbol{x}^{\text{opt}})\right)^\top\boldsymbol{\lambda}^{\text{opt}}$.

(C) $\boldsymbol{u}^{\text{opt}}=-\left(\boldsymbol{F}(\boldsymbol{x}^{\text{opt}})\right)^\top\boldsymbol{\lambda}^{\text{opt}}$. `Reason:` take the mentioned derivative of H from Problem 2.

Problem 5

In problem (1), the quantity $\|\boldsymbol{u}^{\text{opt}}\|_2^2$ must be time-invariant because

(A) $\boldsymbol{u}^{\text{opt}}$ must be time-invariant.

(B) $\boldsymbol{x}^{\text{opt}}$ must be time-invariant.

(C) H^{opt} must be time-invariant. `Reason:` substitute $\boldsymbol{u}^{\text{opt}}$ from Problem 4 in H, and notice that ours is a time-invariant OCP (both $L, \boldsymbol{f}$ have no explicit dependence on t).