Schrödinger Bridges over Kinetic Swarming Models

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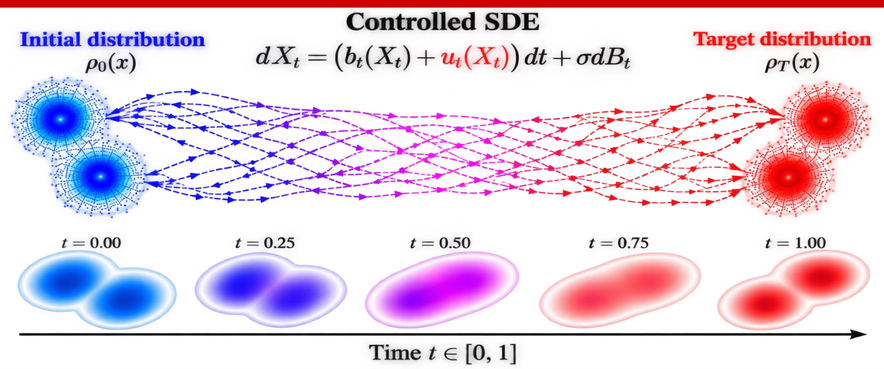
SCHRÖDINGER BRIDGES



KINETIC SWARM MODELS

CUCKER-SMALE KINETICS

$$\begin{cases} \mathrm{d}X_t^i = V_t^i\,\mathrm{d}t, \\ \mathrm{d}V_t^i = \dfrac{1}{N}\displaystyle\sum_{j\neq i,\,j=1}^{N} a\big(\|X_t^i - X_t^j\|\big)\,(V_t^j - V_t^i)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \end{cases}$$

$$a(z) := \frac{K}{(1+z^2)^{\gamma}}, \quad K > 0, \gamma \geq 0.$$

MORSE KINETICS

$$\begin{cases} \mathrm{d}X_t^i = V_t^i\,\mathrm{d}t, \\ \mathrm{d}V_t^i = -\dfrac{1}{N}\displaystyle\sum_{j\neq i}\nabla_{X_t^i} W\big(\|X_t^i - X_t^j\|\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \end{cases}$$

$$W(z) := C_R\,e^{-z/\ell_R} - C_A\,e^{-z/\ell_A},$$

STOCHASTIC OPTIMAL CONTROL PROBLEM

$$\underset{u.,\,\mu.}{\text{Minimize}} \quad \frac{1}{2\sigma^2}\int_0^T\int_{\mathbb{R}^{2d}} \|u_t(x,v)\|^2\,\mu_t\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t,$$

$$\text{subject to} \quad \partial_t\mu_t + \langle v, \nabla_x\mu_t\rangle + \nabla_v\cdot\Big(\mu_t\big(\mathcal{I}_t[\rho_t] + u_t\big)\Big) = \frac{\sigma^2}{2}\Delta_v\mu_t,$$

$$\text{with} \quad \mu_0(x,v) = \mu_{\text{initial}}(x,v), \quad \mu_T(x,v) = \mu_{\text{final}}(x,v).$$

SCHRÖDINGER SYSTEM: CUCKER-SMALE KINETICS

$$\begin{cases} \mu_t^*(x,v) = \varphi_t(x,v)\,\hat{\varphi}_t(x,v), \\ \partial_t\varphi_t + \langle v, \nabla_x\varphi_t\rangle + \langle f_t[\mu_t^*], \nabla_v\varphi_t\rangle + \dfrac{\sigma^2}{2}\Delta_v\varphi_t = r_t(x,v)\,\varphi_t, \\ \partial_t\hat{\varphi}_t + \langle v, \nabla_x\hat{\varphi}_t\rangle + \nabla_v\cdot(\hat{\varphi}_t f_t[\mu_t^*]) - \dfrac{\sigma^2}{2}\Delta_v\hat{\varphi}_t = -r_t(x,v)\,\hat{\varphi}_t, \end{cases}$$

where

$$f_t[\mu_t^*](x,v) = \int_{\mathbb{R}^{2d}} a(\|x-\tilde{x}\|)\,\mu_t^*(\tilde{x},\tilde{v})\,(\tilde{v}-v)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v},$$

$$r_t(x,v) = \int_{\mathbb{R}^{2d}} a(\|x-\tilde{x}\|)\,\hat{\varphi}_t(\tilde{x},\tilde{v})\,\langle\tilde{v}-v, \nabla_{\tilde{v}}\varphi_t(\tilde{x},\tilde{v})\rangle\,d\tilde{x}\,d\tilde{v},$$

subject to

$$\varphi_0(x,v)\hat{\varphi}_0(x,v) = \mu_{\text{initial}}(x,v),$$
$$\varphi_T(x,v)\hat{\varphi}_T(x,v) = \mu_{\text{final}}(x,v).$$

SCHRÖDINGER SYSTEM: MORSE KINETICS

$$\begin{cases} \mu_t^*(x,v) = \varphi_t(x,v)\,\hat{\varphi}_t(x,v), \\ \partial_t\varphi_t + \langle v, \nabla_x\varphi_t\rangle + \langle g_t[\rho_t^*], \nabla_v\varphi_t\rangle + \dfrac{\sigma^2}{2}\Delta_v\varphi_t = s_t(x)\,\varphi_t, \\ \partial_t\hat{\varphi}_t + \langle v, \nabla_x\hat{\varphi}_t\rangle + \langle g_t[\rho_t^*], \nabla_v\hat{\varphi}_t\rangle - \dfrac{\sigma^2}{2}\Delta_v\hat{\varphi}_t = -s_t(x)\,\hat{\varphi}_t, \end{cases}$$

where

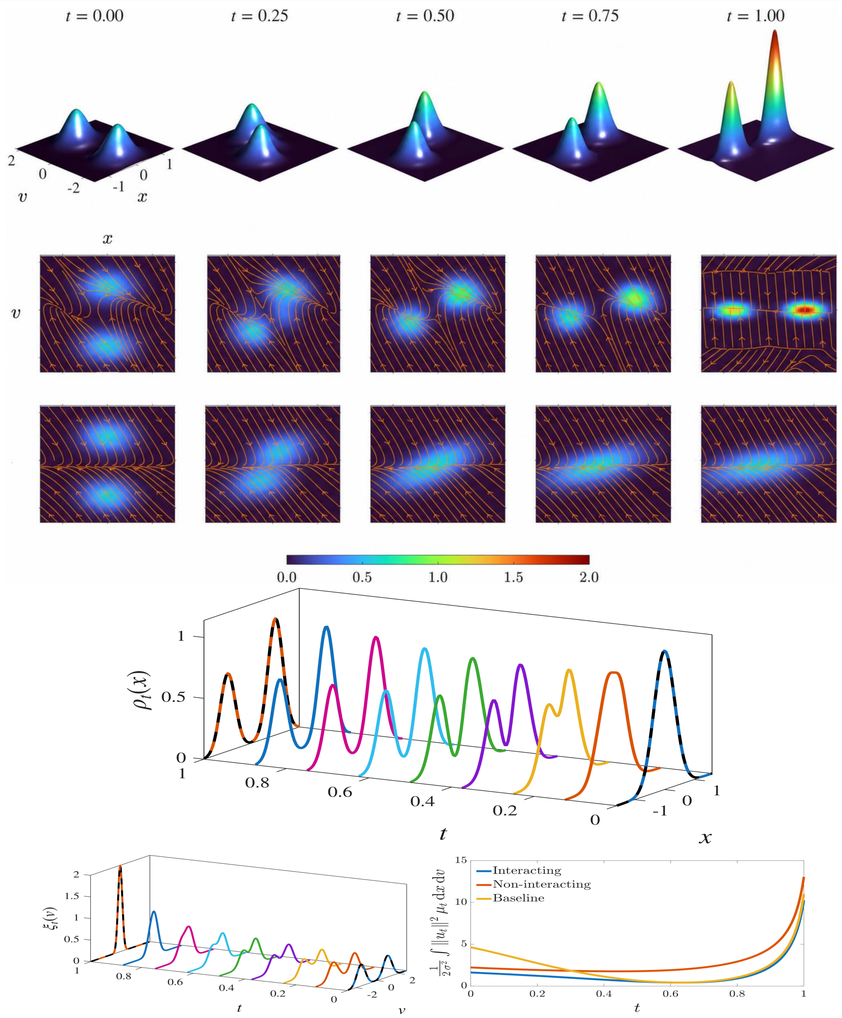
$$g_t[\rho_t^*](x) = -\int_{\mathbb{R}^d}\nabla_x W(\|x-\tilde{x}\|)\,\rho_t^*(\tilde{x})\,\mathrm{d}\tilde{x},$$

$$s_t(x) := -\int_{\mathbb{R}^{2d}}\big\langle\nabla_x W(\|x-\tilde{x}\|), \nabla_{\tilde{v}}\varphi_t(\tilde{x},\tilde{v})\big\rangle\,\hat{\varphi}_t(\tilde{x},\tilde{v})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{v},$$

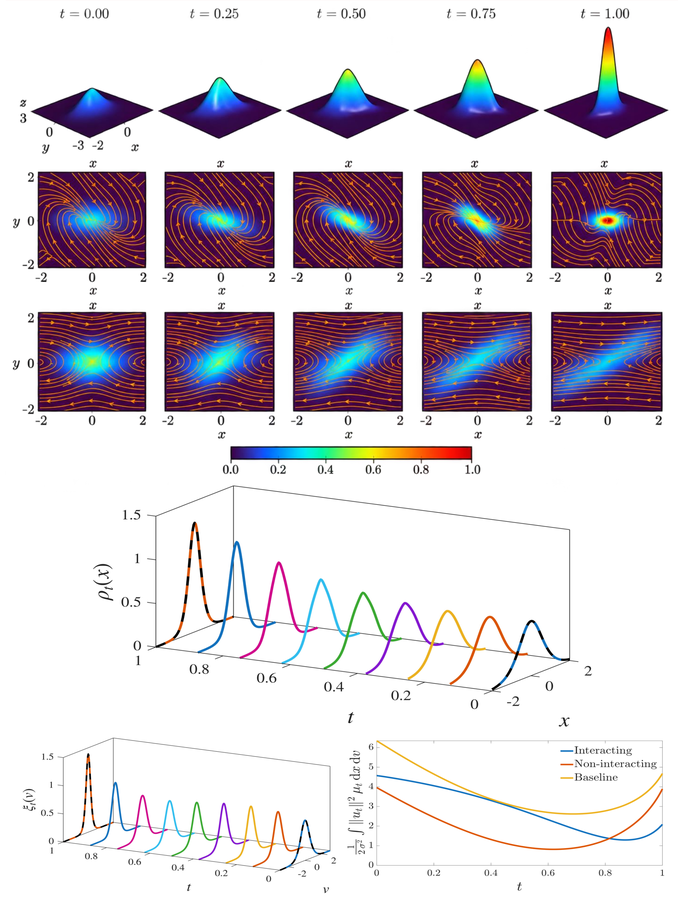
subject to

$$\varphi_0(x,v)\hat{\varphi}_0(x,v) = \mu_{\text{initial}}(x,v),$$
$$\varphi_T(x,v)\hat{\varphi}_T(x,v) = \mu_{\text{final}}(x,v).$$

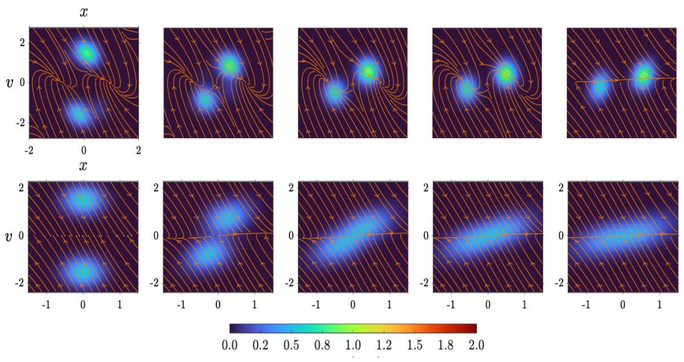
CUCKER-SMALE MFSB



MORSE MFSB



MFSB WITH PARTIAL STATE INFORMATION



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