Fixed Horizon Linear Quadratic Optimal Covariance Steering in Continuous Time with Hilbert-Schmidt Terminal Cost

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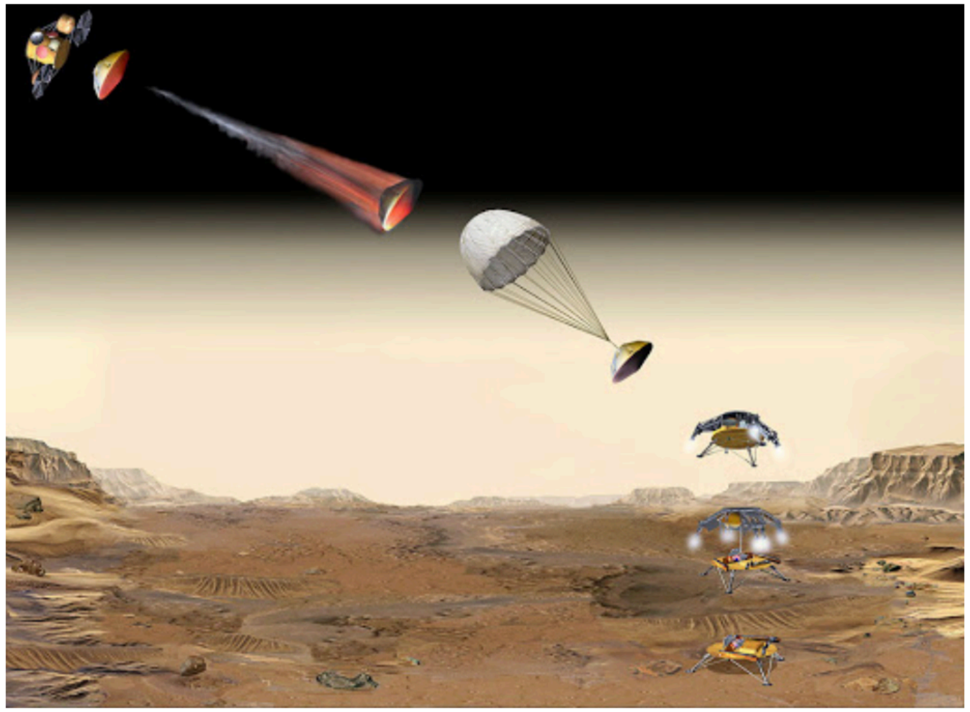
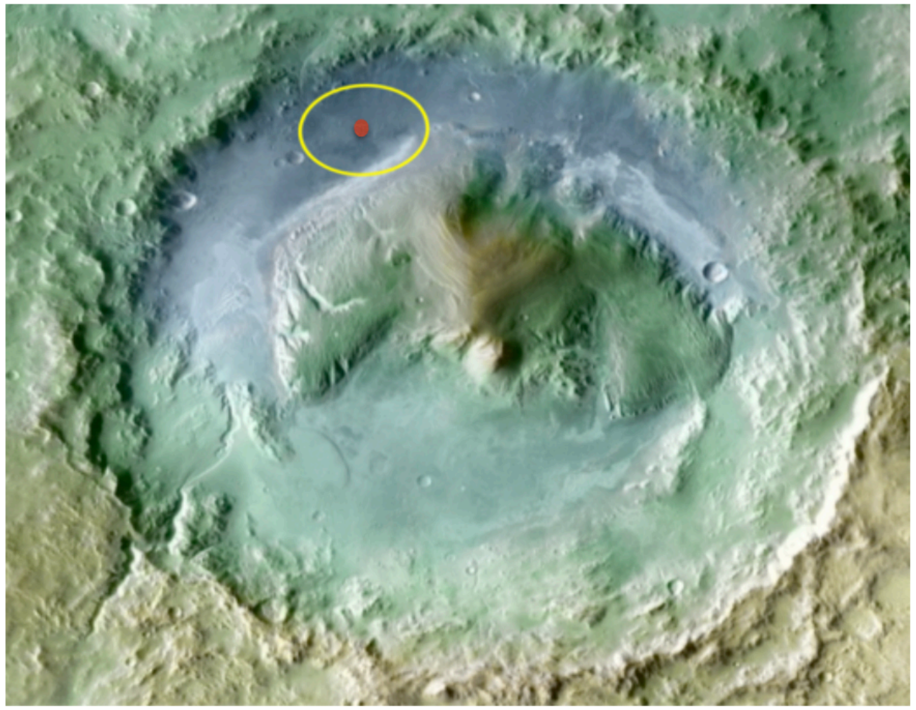
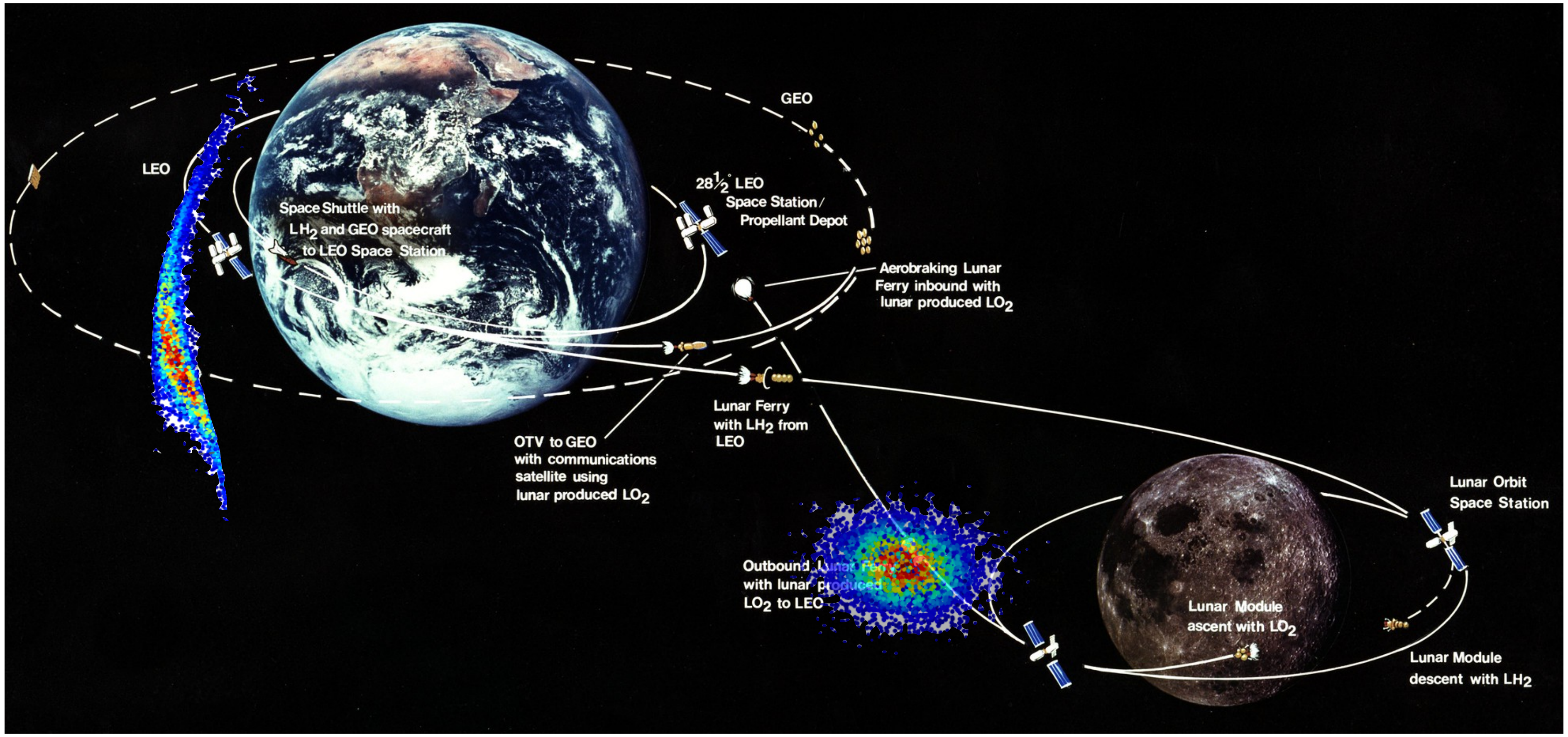
Motivation: Control of Probability Distributions

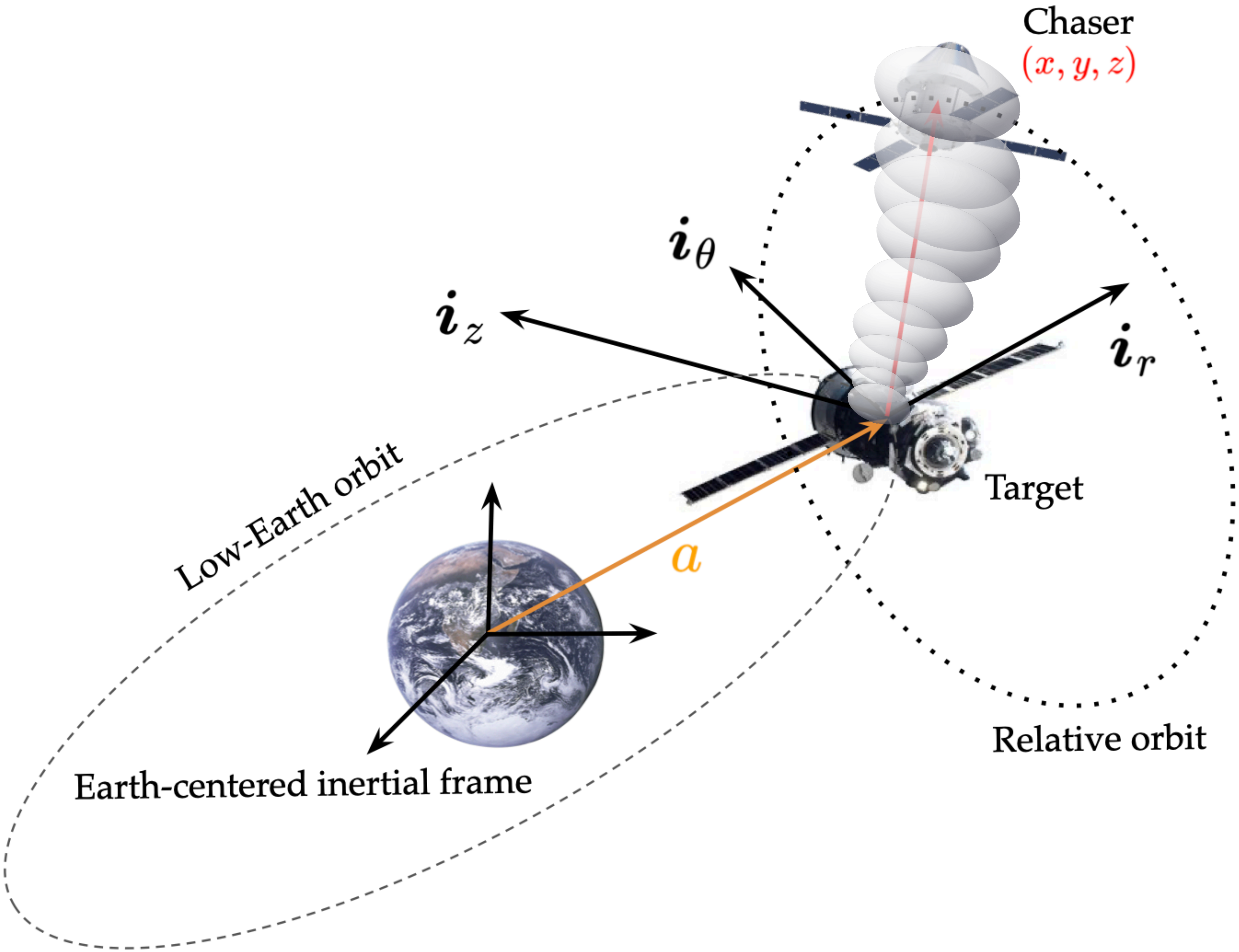




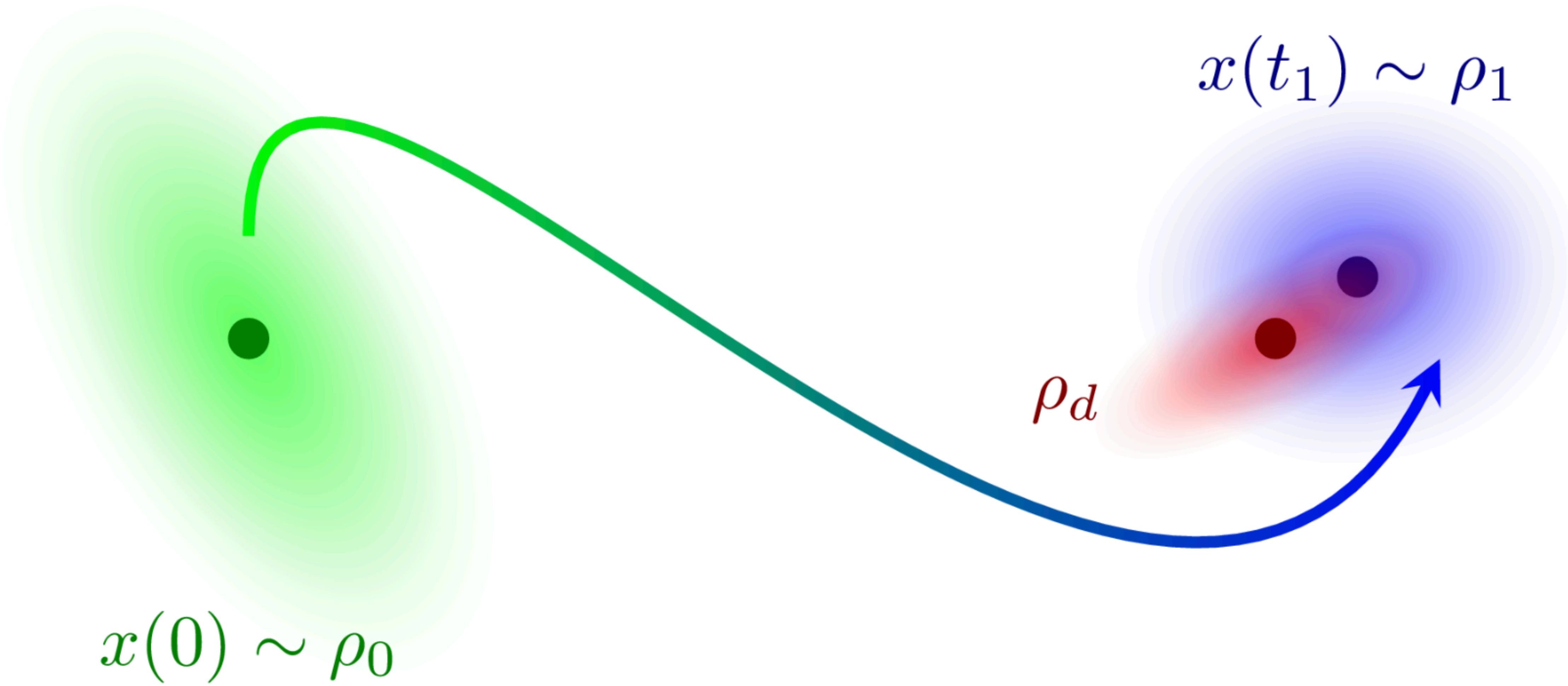
Gale Crater (4.49S, 137.42E)



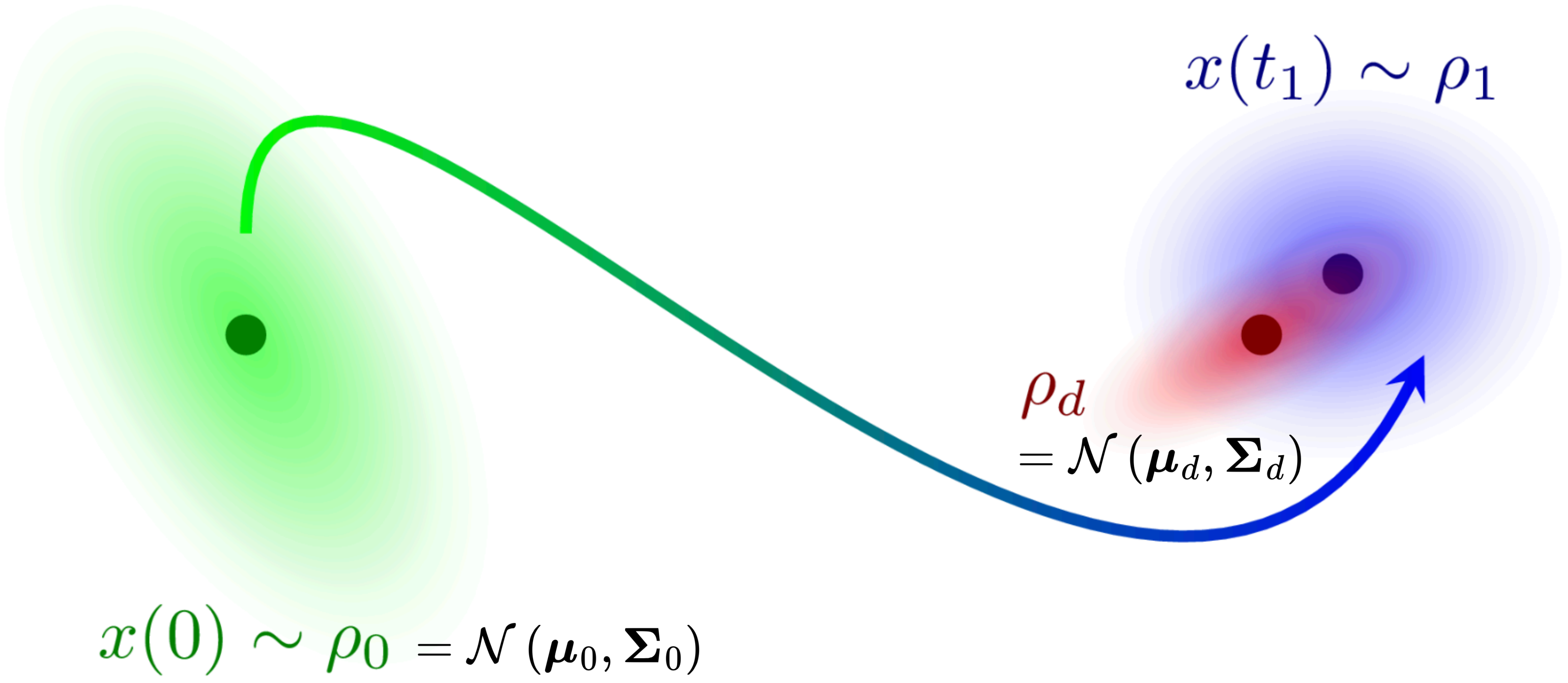
Motivation: Control of Probability Distributions



This Work: Optimal Covariance Control



This Work: Optimal Covariance Control



Current State-of-the-art

Well-investigated in **discrete time**

Balci and Bakolas, LCSS 2020

Balci, Halder and Bakolas, CDC 2021

Yin et al., ICRA 2022

Saravanos et al., IROS 2024

Morimoto and Kashima, LCSS 2024

Less investigated in **continuous time**

Halder and Wendel, ACC 2016

- derived matrix BVP for LQ case with optimal transport terminal cost + shooting method
- design of custom algorithm remains open

Hoshino, CDC 2023

- derived FBSDE for nonlinear non-Gaussian case with optimal transport terminal cost
- did not investigate numerical solution

LQ Optimal Control Problem Formulation

Linear controlled dynamics:

$$\mathrm{d}\boldsymbol{x}_t = \boldsymbol{A}_t \underbrace{\boldsymbol{x}_t}_{\text{State}} \,\mathrm{d}t + \boldsymbol{B}_t \underbrace{\boldsymbol{u}_t}_{\text{Input}} \,\mathrm{d}t + \boldsymbol{B}_t \,\mathrm{d}\underbrace{\boldsymbol{w}_t}_{\text{Process noise}}$$

Average quadratic cost-to-go:

$$\inf_{\boldsymbol{u}_t \in \mathcal{U}} \underbrace{\phi\left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d\right)}_{\text{Terminal cost}} + \underbrace{\int_{t_0}^{t_1} \mathbb{E}\left(\|\boldsymbol{u}_t\|_2^2 + \boldsymbol{x}_t^\top \boldsymbol{Q}_t \boldsymbol{x}_t\right) \mathrm{d}t}_{\text{Average cost-to-go}}$$

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Standing assumptions:

$(\boldsymbol{A}_t, \boldsymbol{B}_t) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is uniformly controllable, bounded and continuous w.r.t. time t

$$\boldsymbol{Q}_t \succeq \boldsymbol{0} \;\forall t \in [t_0, t_1],\; \boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_d \succ \boldsymbol{0}$$

Choice of the Terminal Cost $\phi\left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d\right)$

This work: Squared Frobenius a.k.a. Hilbert-Schmidt norm

$$\frac{1}{2}\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_d\|_2^2 + \frac{1}{2}\|\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_d\|_{\text{Frobenius}}^2$$

Prior work: Squared Wasserstein a.k.a. Optimal Transport Cost

$$\frac{1}{2}\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_d\|_2^2 + \frac{1}{2}\text{trace}\left(\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_d - 2\left(\boldsymbol{\Sigma}_1^{\frac{1}{2}}\boldsymbol{\Sigma}_d\boldsymbol{\Sigma}_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)$$

Halder and Wendel, ACC 2016

Recap

Frobenius a.k.a. Hilbert-Schmidt **inner product**

$$\left\langle {\color{red}\blacksquare}, {\color{blue}\blacksquare} \right\rangle := \mathrm{trace}\left({\color{blue}\blacksquare}^{\top} {\color{red}\blacksquare}\right) = \sum_{i,j} {\color{blue}\blacksquare}_{ij} {\color{red}\blacksquare}_{ij}$$

Norm induced by this inner product

$$\left\langle {\color{red}\blacksquare}, {\color{red}\blacksquare} \right\rangle = \left\| {\color{red}\blacksquare} \right\|^2$$

Generalizes squared Euclidean distance between vectors

The Optimal Controller with Our Terminal Cost

From standard LQ theory: $$\boldsymbol{u}_t^{\text{opt}} = \underbrace{\boldsymbol{K}_t^{\text{opt}} \boldsymbol{x}_t}_{\text{Feedback}} + \underbrace{\boldsymbol{v}_t^{\text{opt}}}_{\text{Feedforward}}$$

The Optimal Controller with Our Terminal Cost

From standard LQ theory: $$\boldsymbol{u}_t^{\mathrm{opt}} = \underbrace{\boldsymbol{K}_t^{\mathrm{opt}} \boldsymbol{x}_t}_{\text{Feedback}} + \underbrace{\boldsymbol{v}_t^{\mathrm{opt}}}_{\text{Feedforward}}$$

Decoupling of mean and covariance control:

Feedforward gain

$$\boldsymbol{v}_t^{\mathrm{opt}} = \boldsymbol{B}_t^{\top} \left(\boldsymbol{P}_t^{\mathrm{opt}} \boldsymbol{\mu}_t^{\mathrm{opt}} - \boldsymbol{z}_t^{\mathrm{opt}} \right)$$

$$\begin{pmatrix} \dot{\boldsymbol{\mu}}_t^{\mathrm{opt}} \\ \dot{\boldsymbol{z}}_t^{\mathrm{opt}} \end{pmatrix} = \begin{bmatrix} \boldsymbol{A}_t & -\boldsymbol{B}_t \boldsymbol{B}_t^{\top} \\ -\boldsymbol{Q}_t & -\boldsymbol{A}_t^{\top} \end{bmatrix} \begin{pmatrix} \boldsymbol{\mu}_t^{\mathrm{opt}} \\ \boldsymbol{z}_t^{\mathrm{opt}} \end{pmatrix}$$

$$\boldsymbol{\mu}_0 \in \mathbb{R}^n \text{ given}, \quad \boldsymbol{z}_1^{\mathrm{opt}} = \frac{\partial \phi}{\partial \boldsymbol{\mu}_1^{\mathrm{opt}}} = \boldsymbol{\mu}_1^{\mathrm{opt}} - \boldsymbol{\mu}_d$$

$$\boldsymbol{K}_t^{\mathrm{opt}} = -\boldsymbol{B}_t^{\top} \boldsymbol{P}_t^{\mathrm{opt}}$$ Feedback gain

$$\dot{\boldsymbol{\Sigma}}_t^{\mathrm{opt}} = \left(\boldsymbol{A}_t - \boldsymbol{B}_t \boldsymbol{B}_t^{\top} \boldsymbol{P}_t^{\mathrm{opt}} \right) \boldsymbol{\Sigma}_t^{\mathrm{opt}} + \boldsymbol{\Sigma}_t^{\mathrm{opt}} \left(\boldsymbol{A}_t - \boldsymbol{B}_t \boldsymbol{B}_t^{\top} \boldsymbol{P}_t^{\mathrm{opt}} \right)^{\top} + \boldsymbol{B}_t \boldsymbol{B}_t^{\top}$$

$$-\dot{\boldsymbol{P}}_t^{\mathrm{opt}} = \boldsymbol{A}_t^{\top} \boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{P}_t^{\mathrm{opt}} \boldsymbol{A}_t - \boldsymbol{P}_t^{\mathrm{opt}} \boldsymbol{B}_t \boldsymbol{B}_t^{\top} \boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{Q}_t$$

$$\boldsymbol{\Sigma}_0 \succ \boldsymbol{0} \text{ given},$$

$$\boldsymbol{P}_1^{\mathrm{opt}} = \boldsymbol{\Sigma}_1^{\mathrm{opt}} - \boldsymbol{\Sigma}_d, \ \boldsymbol{\Sigma}_d \succ \boldsymbol{0} \text{ given}.$$

Rest of this Talk

Focus on covariance control design:

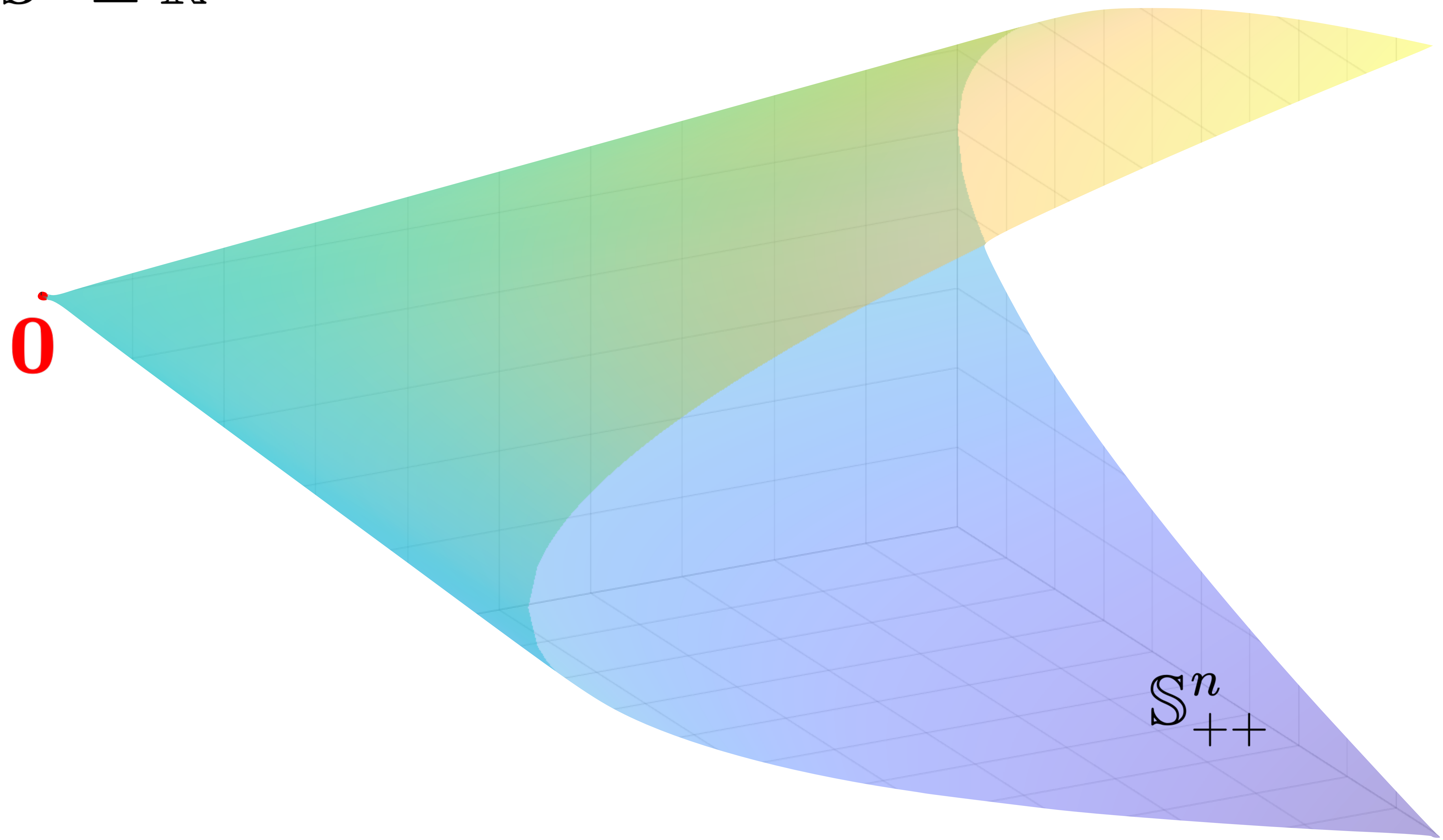
$$
\begin{aligned}
\boldsymbol{K}_t^{\mathrm{opt}} &= -\boldsymbol{B}_t^\top \boldsymbol{P}_t^{\mathrm{opt}} \quad \text{Feedback gain}\\
\dot{\boldsymbol{\Sigma}}_t^{\mathrm{opt}} &= \left(\boldsymbol{A}_t - \boldsymbol{B}_t\boldsymbol{B}_t^\top\boldsymbol{P}_t^{\mathrm{opt}}\right)\boldsymbol{\Sigma}_t^{\mathrm{opt}} + \boldsymbol{\Sigma}_t^{\mathrm{opt}}\left(\boldsymbol{A}_t - \boldsymbol{B}_t\boldsymbol{B}_t^\top\boldsymbol{P}_t^{\mathrm{opt}}\right)^\top + \boldsymbol{B}_t\boldsymbol{B}_t^\top\\
-\dot{\boldsymbol{P}}_t^{\mathrm{opt}} &= \boldsymbol{A}_t^\top\boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{P}_t^{\mathrm{opt}}\boldsymbol{A}_t - \boldsymbol{P}_t^{\mathrm{opt}}\boldsymbol{B}_t\boldsymbol{B}_t^\top\boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{Q}_t\\
\boldsymbol{\Sigma}_0 &\succ \boldsymbol{0}\,\text{given},\\
\boldsymbol{P}_1^{\mathrm{opt}} &= \boldsymbol{\Sigma}_1^{\mathrm{opt}} - \boldsymbol{\Sigma}_d,\ \boldsymbol{\Sigma}_d \succ \boldsymbol{0}\,\text{given}.
\end{aligned}
$$

Coupled Lyapunov-Riccati nonlinear matrix ODE BVP

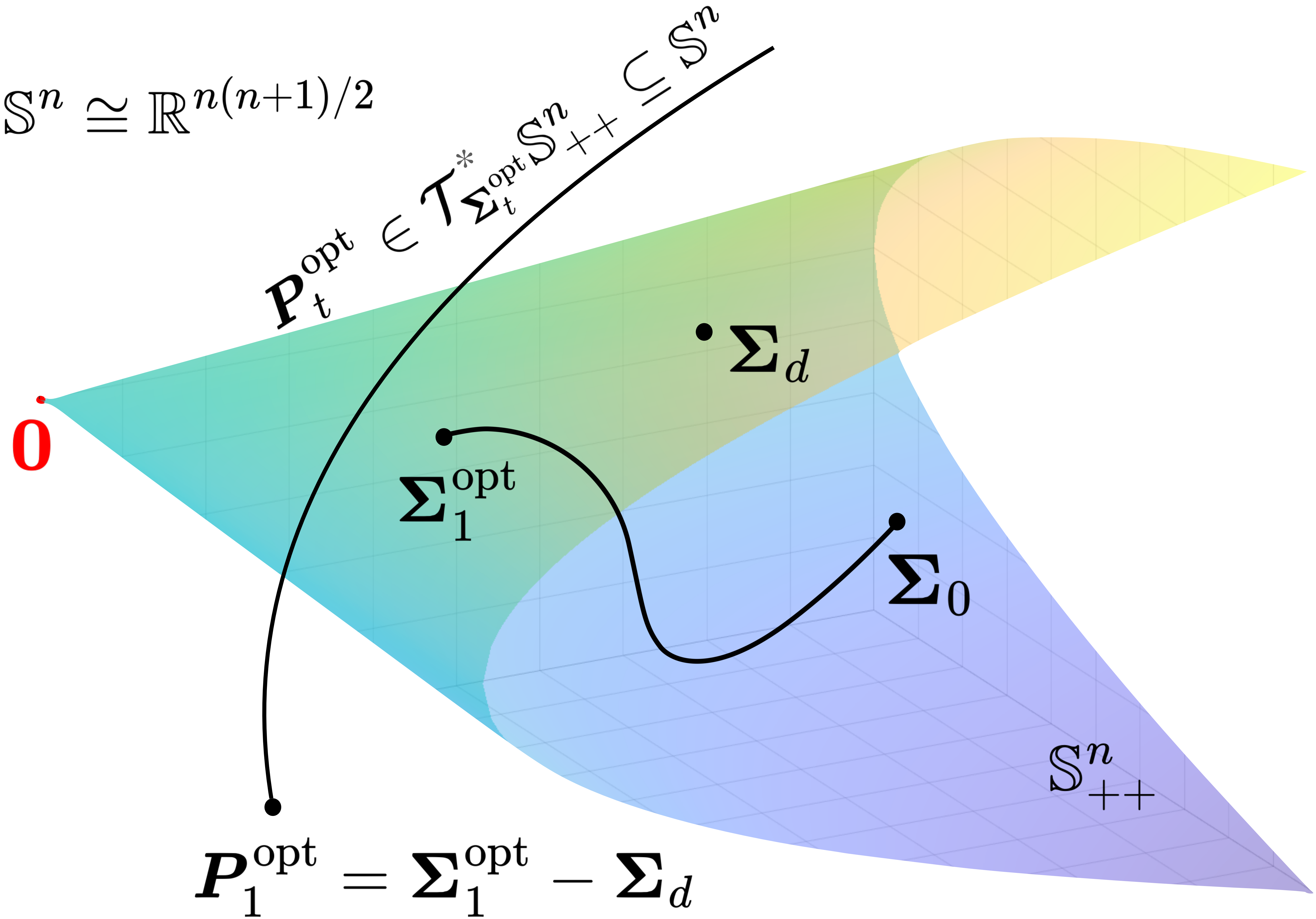
Our contribution: design numerical algorithm with guarantees to solve this system

Geometry of Our Matrix ODE BVP

$\mathbb{S}^n \cong \mathbb{R}^{n(n+1)/2}$



Geometry of Our Matrix ODE BVP



Idea

Change of variable $\boldsymbol{\Sigma}_t \mapsto \boldsymbol{H}_t$

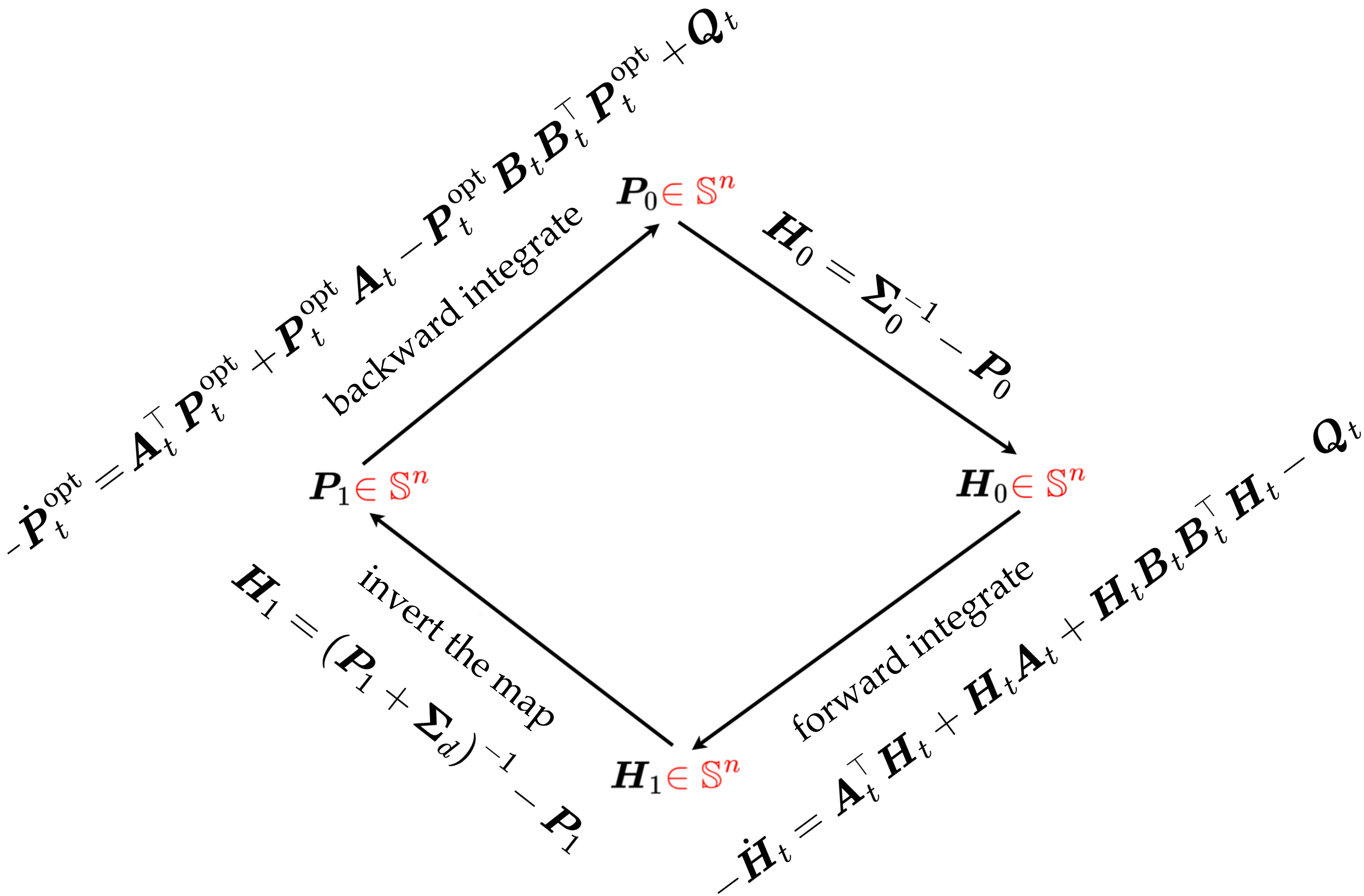
$$\boxed{\boldsymbol{H}_t := \boldsymbol{\Sigma}_t^{-1} - \boldsymbol{P}_t}$$

From Lyapunov-Riccati to Riccati-Riccati system:

$$\begin{aligned}
-\dot{\boldsymbol{P}}_t^{\mathrm{opt}} &= \boldsymbol{A}_t^\top \boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{P}_t^{\mathrm{opt}} \boldsymbol{A}_t - \boldsymbol{P}_t^{\mathrm{opt}} \boldsymbol{B}_t \boldsymbol{B}_t^\top \boldsymbol{P}_t^{\mathrm{opt}} + \boldsymbol{Q}_t \\
-\dot{\boldsymbol{H}}_t &= \boldsymbol{A}_t^\top \boldsymbol{H}_t + \boldsymbol{H}_t \boldsymbol{A}_t + \boldsymbol{H}_t \boldsymbol{B}_t \boldsymbol{B}_t^\top \boldsymbol{H}_t - \boldsymbol{Q}_t \\
\boldsymbol{H}_0 &= \boldsymbol{\Sigma}_0^{-1} - \boldsymbol{P}_0 \\
\boldsymbol{H}_1 &= (\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)^{-1} - \boldsymbol{P}_1
\end{aligned}$$

Achieves ODE-level decoupling but boundary conditions remain coupled

Proposed Fixed Point Recursion: First Cut



Proposed Fixed Point Recursion: First Cut

Can it be replaced by an algebraic map?

$$-\dot{\boldsymbol{P}}_t^{\text{opt}} = \boldsymbol{A}_t^\top \boldsymbol{P}_t^{\text{opt}} + \boldsymbol{P}_t^{\text{opt}} \boldsymbol{A}_t - \boldsymbol{P}_t^{\text{opt}} \boldsymbol{B}_t \boldsymbol{B}_t^\top \boldsymbol{P}_t^{\text{opt}} + \boldsymbol{Q}_t$$

$$\begin{tikzcd}
& \boldsymbol{P}_0 \in \mathbb{S}^n \arrow[dr, "\boldsymbol{H}_0 = \boldsymbol{\Sigma}_0^{-1} - \boldsymbol{P}_0"] & \\
\boldsymbol{P}_1 \in \mathbb{S}^n \arrow[ur, "\text{backward integrate}"] & & \boldsymbol{H}_0 \in \mathbb{S}^n \arrow[dl, "\text{forward integrate}"] \\
& \boldsymbol{H}_1 \in \mathbb{S}^n \arrow[ul, "\text{invert the map}", "\boldsymbol{H}_1 = (\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)^{-1} - \boldsymbol{P}_1"'] &
\end{tikzcd}$$

Multi-valued symmetric inverse

Which solution to use?

$$-\dot{\boldsymbol{H}}_t = \boldsymbol{A}_t^\top \boldsymbol{H}_t + \boldsymbol{H}_t \boldsymbol{A}_t + \boldsymbol{H}_t \boldsymbol{B}_t \boldsymbol{B}_t^\top \boldsymbol{H}_t - \boldsymbol{Q}_t$$

Can it be replaced by an algebraic map?

Mappings $P_1 \mapsto P_0$, $H_0 \mapsto H_1$

Let $\boldsymbol{M}_t := \begin{bmatrix} \boldsymbol{A}_t & -\boldsymbol{B}_t\boldsymbol{B}_t^\top \\ -\boldsymbol{Q}_t & -\boldsymbol{A}_t^\top \end{bmatrix}$

State transition matrix for this $2n \times 2n$ Hamiltonian matrix:

$$\partial_t \boldsymbol{\Phi}(s,t) = \boldsymbol{M}(t)\boldsymbol{\Phi}(s,t), \quad \boldsymbol{\Phi}(s,s) = \boldsymbol{I}$$

Consider mild abuse of notation: $\begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\Phi}_{11}(t_0,t_1) & \boldsymbol{\Phi}_{12}(t_0,t_1) \\ \boldsymbol{\Phi}_{21}(t_0,t_1) & \boldsymbol{\Phi}_{22}(t_0,t_1) \end{bmatrix}$

Thm. Desired maps have bijective linear fractional transform representations:

$$\boldsymbol{P}_0 = \left(\boldsymbol{P}_1\boldsymbol{\Phi}_{12} - \boldsymbol{\Phi}_{22}\right)^{-1}\left(\boldsymbol{\Phi}_{21} - \boldsymbol{P}_1\boldsymbol{\Phi}_{11}\right)$$

$$\boldsymbol{H}_1 = -\left(\boldsymbol{\Phi}_{11}^\top - \boldsymbol{H}_0\boldsymbol{\Phi}_{12}^\top\right)^{-1}\left(\boldsymbol{\Phi}_{21}^\top - \boldsymbol{H}_0\boldsymbol{\Phi}_{22}^\top\right)$$

Mapping $\boldsymbol{H}_1 \mapsto \boldsymbol{P}_1$

We have

$$\boldsymbol{H}_1 = (\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)^{-1} - \boldsymbol{P}_1$$

This is equivalent to either of

$$\boldsymbol{I} = \boldsymbol{\Sigma}_1 (\boldsymbol{P}_1 + \boldsymbol{H}_1) = (\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)(\boldsymbol{P}_1 + \boldsymbol{H}_1)$$
$$\boldsymbol{I} = (\boldsymbol{P}_1 + \boldsymbol{H}_1) \boldsymbol{\Sigma}_1 = (\boldsymbol{P}_1 + \boldsymbol{H}_1)(\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)$$

Which is equivalent to either of

$$\boldsymbol{P}_1^2 + \boldsymbol{P}_1 \boldsymbol{H}_1 + \boldsymbol{\Sigma}_d \boldsymbol{P}_1 + (\boldsymbol{\Sigma}_d \boldsymbol{H}_1 - \boldsymbol{I}) = \boldsymbol{0}$$
$$\boldsymbol{P}_1^2 + \boldsymbol{P}_1 \boldsymbol{\Sigma}_d + \boldsymbol{H}_1 \boldsymbol{P}_1 + (\boldsymbol{H}_1 \boldsymbol{\Sigma}_d - \boldsymbol{I}) = \boldsymbol{0}$$

Effectively one equation in one symmetric unknown $\boldsymbol{P}_1$

But which of them or **equivalent of them** is suitable for our recursion?

Mapping $\boldsymbol{H}_1 \mapsto \boldsymbol{P}_1$

One natural equivalent is linear (Sylvester's equation):

$$(\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d)\boldsymbol{P}_1 + \boldsymbol{P}_1(\boldsymbol{\Sigma}_d - \boldsymbol{H}_1) = \boldsymbol{\Sigma}_d\boldsymbol{H}_1 - \boldsymbol{H}_1\boldsymbol{\Sigma}_d$$

Thm. Given $\boldsymbol{H}_1 \in \mathbb{S}^n, \boldsymbol{\Sigma}_d \in \mathbb{S}^n_{++},$ there exists non-unique $\boldsymbol{P}_1 \in \mathbb{S}^n$ solving above

Mapping $\boldsymbol{H}_1 \mapsto \boldsymbol{P}_1$

One natural equivalent is linear (Sylvester's equation):

$$(\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d)\boldsymbol{P}_1 + \boldsymbol{P}_1(\boldsymbol{\Sigma}_d - \boldsymbol{H}_1) = \boldsymbol{\Sigma}_d\boldsymbol{H}_1 - \boldsymbol{H}_1\boldsymbol{\Sigma}_d$$

Thm. Given $\boldsymbol{H}_1 \in \mathbb{S}^n, \boldsymbol{\Sigma}_d \in \mathbb{S}^n_{++},$ there exists non-unique $\boldsymbol{P}_1 \in \mathbb{S}^n$ solving above

Valid candidate solutions:

$-\boldsymbol{H}_1, -\boldsymbol{\Sigma}_d,$ solution of the linear system:

$$((\boldsymbol{I} \otimes (\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d)) - ((\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d) \otimes \boldsymbol{I}))\mathrm{vec}(\boldsymbol{P}_1) = \mathrm{vec}(\boldsymbol{\Sigma}_d\boldsymbol{H}_1 - \boldsymbol{H}_1\boldsymbol{\Sigma}_d)$$

But for these solutions, $(\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d)^{-1}$ may not exist

They also fail to make proposed recursion converge!!

Mapping $\boldsymbol{H}_1 \mapsto \boldsymbol{P}_1$

Other natural equivalent is the quadratic equation:

$$\boldsymbol{P}_1^2 + \boldsymbol{P}_1\left(\frac{\boldsymbol{H}_1 + \boldsymbol{\Sigma}_d}{2}\right) + \left(\frac{\boldsymbol{H}_1 + \boldsymbol{\Sigma}_d}{2}\right)\boldsymbol{P}_1 + \left(\frac{\boldsymbol{\Sigma}_d \boldsymbol{H}_1 + \boldsymbol{H}_1 \boldsymbol{\Sigma}_d}{2} - \boldsymbol{I}\right) = \boldsymbol{0}$$

This is a special instance of Continuous-time Algebraic Riccati Equation (CARE)

Thm. Given $\boldsymbol{H}_1 \in \mathbb{S}^n, \boldsymbol{\Sigma}_d \in \mathbb{S}^n_{++},$ there exists non-unique $\boldsymbol{P}_1 \in \mathbb{S}^n$ solving above. Among them, the only stabilizing solution is

$$\boldsymbol{P}_1 \equiv \boldsymbol{P}_1^{\text{stab}} = -\frac{\boldsymbol{H}_1 + \boldsymbol{\Sigma}_d}{2} + \left(\left(\frac{\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d}{2}\right)^2 + \boldsymbol{I}\right)^{\frac{1}{2}}$$

We prove that for this solution $\left(\boldsymbol{P}_1 + \boldsymbol{\Sigma}_d\right)^{-1}$ exists

We also prove that this solution makes the proposed recursion converge!!

Mapping $\boldsymbol{H}_1 \mapsto \boldsymbol{P}_1$

Stabilizing (resp. anti-stabilizing) solution for CARE

$$\boldsymbol{P}_1 \bar{\boldsymbol{B}} \boldsymbol{P}_1 + \boldsymbol{P}_1 \bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^\top \boldsymbol{P}_1 - \bar{\boldsymbol{C}} = \boldsymbol{0}$$

is one that makes the closed-loop spectrum

$$\text{spectrum}\left(-\bar{\boldsymbol{A}} - \bar{\boldsymbol{B}} \bar{\boldsymbol{B}}^\top \boldsymbol{P}_1\right) \subset \mathbb{C}^- \quad \left(\text{resp. } \mathbb{C}^+\right)$$

All solutions $\boldsymbol{P}_1 \in \mathbb{S}^n$ of our CARE satisfy the conic ordering:

$$\boldsymbol{P}_1^{\text{antistab}} \preceq \boldsymbol{P}_1 \preceq \boldsymbol{P}_1^{\text{stab}}$$

In general, $\boldsymbol{P}_1^{\text{antistab}}, \boldsymbol{P}_1^{\text{stab}}$ are sign-indefinite

Proposed Fixed Point Recursion: Final Cut

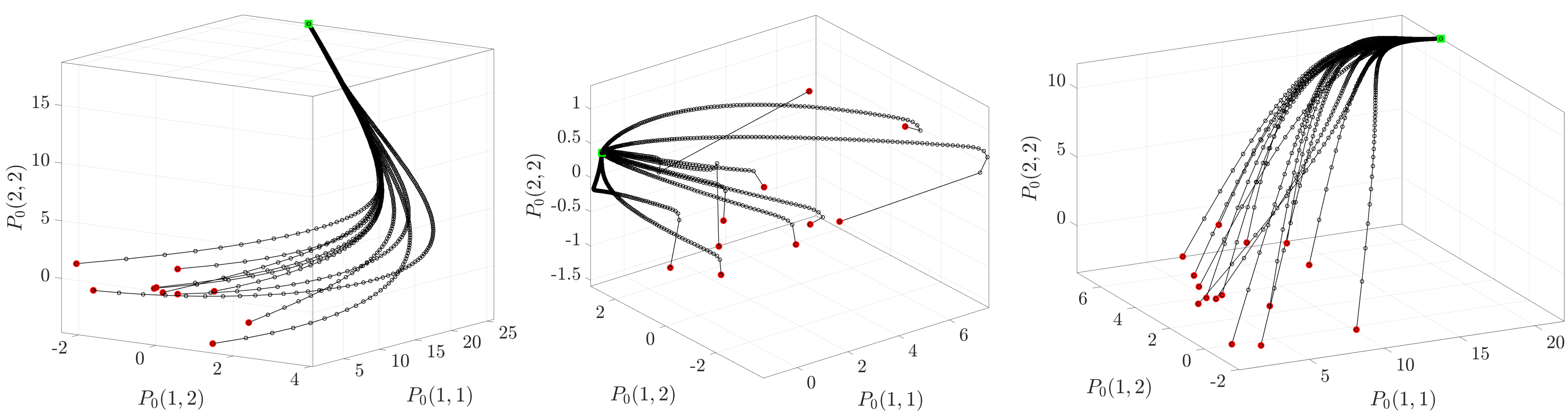
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
& \boldsymbol{P}_0 \in \mathbb{S}^n \arrow[dr, "\boldsymbol{\Sigma}_0^{-1} - \boldsymbol{P}_0"] & \\
\boldsymbol{P}_1 \in \mathbb{S}^n \arrow[ur, "{\text{linear fractional transform}\ \left(\boldsymbol{P}_1\boldsymbol{\Phi}_{12} - \boldsymbol{\Phi}_{22}\right)^{-1}\left(\boldsymbol{\Phi}_{21} - \boldsymbol{P}_1\boldsymbol{\Phi}_{11}\right)}"] & & \boldsymbol{H}_0 \in \mathbb{S}^n \arrow[dl, "{\text{linear fractional transform}\ -\left(\boldsymbol{\Phi}_{11}^\top - \boldsymbol{H}_0\boldsymbol{\Phi}_{12}^\top\right)^{-1}\left(\boldsymbol{\Phi}_{21}^\top - \boldsymbol{H}_0\boldsymbol{\Phi}_{22}^\top\right)}"] \\
& \boldsymbol{H}_1 \in \mathbb{S}^n \arrow[ul, "{-\frac{\boldsymbol{H}_1 + \boldsymbol{\Sigma}_d}{2} + \left(\left(\frac{\boldsymbol{H}_1 - \boldsymbol{\Sigma}_d}{2}\right)^2 + \boldsymbol{I}\right)^{\frac{1}{2}}}"] &
\end{tikzcd}
$$

Proposed Fixed Point Recursion: Convergence

Thm. Let
$$\boldsymbol{F}_1\left(\boldsymbol{X}\right) := \boldsymbol{\Sigma}_0^{-1} - \boldsymbol{X},$$
$$\boldsymbol{F}_2\left(\boldsymbol{X}\right) := -\left(\boldsymbol{\Phi}_{11}^{\top} - \boldsymbol{X}\boldsymbol{\Phi}_{12}^{\top}\right)^{-1}\left(\boldsymbol{\Phi}_{21}^{\top} - \boldsymbol{X}\boldsymbol{\Phi}_{22}^{\top}\right)$$
$$\boldsymbol{F}_3\left(\boldsymbol{X}\right) := -\frac{\boldsymbol{X}+\boldsymbol{\Sigma}_d}{2} + \left(\left(\frac{\boldsymbol{X}-\boldsymbol{\Sigma}_d}{2}\right)^2 + \boldsymbol{I}\right)^{\frac{1}{2}}$$
$$\boldsymbol{F}_4\left(\boldsymbol{X}\right) := \left(\boldsymbol{X}\boldsymbol{\Phi}_{12} - \boldsymbol{\Phi}_{22}\right)^{-1}\left(\boldsymbol{\Phi}_{21} - \boldsymbol{X}\boldsymbol{\Phi}_{11}\right),$$

in variable $\boldsymbol{X} \in \mathbb{S}^n$. Then $\boldsymbol{F} := \boldsymbol{F}_4 \circ \boldsymbol{F}_3 \circ \boldsymbol{F}_2 \circ \boldsymbol{F}_1$ has unique fixed point in $\mathbb{S}^n$.

Also, the recursion $\boldsymbol{P}_0 \mapsto \left(\boldsymbol{P}_0\right)_{next} = \boldsymbol{F}\left(\boldsymbol{P}_0\right)$ converges to it for a.e. $\boldsymbol{P}_0 \in \mathbb{S}^n$.



Numerical Case Study: Noisy Double Integrator

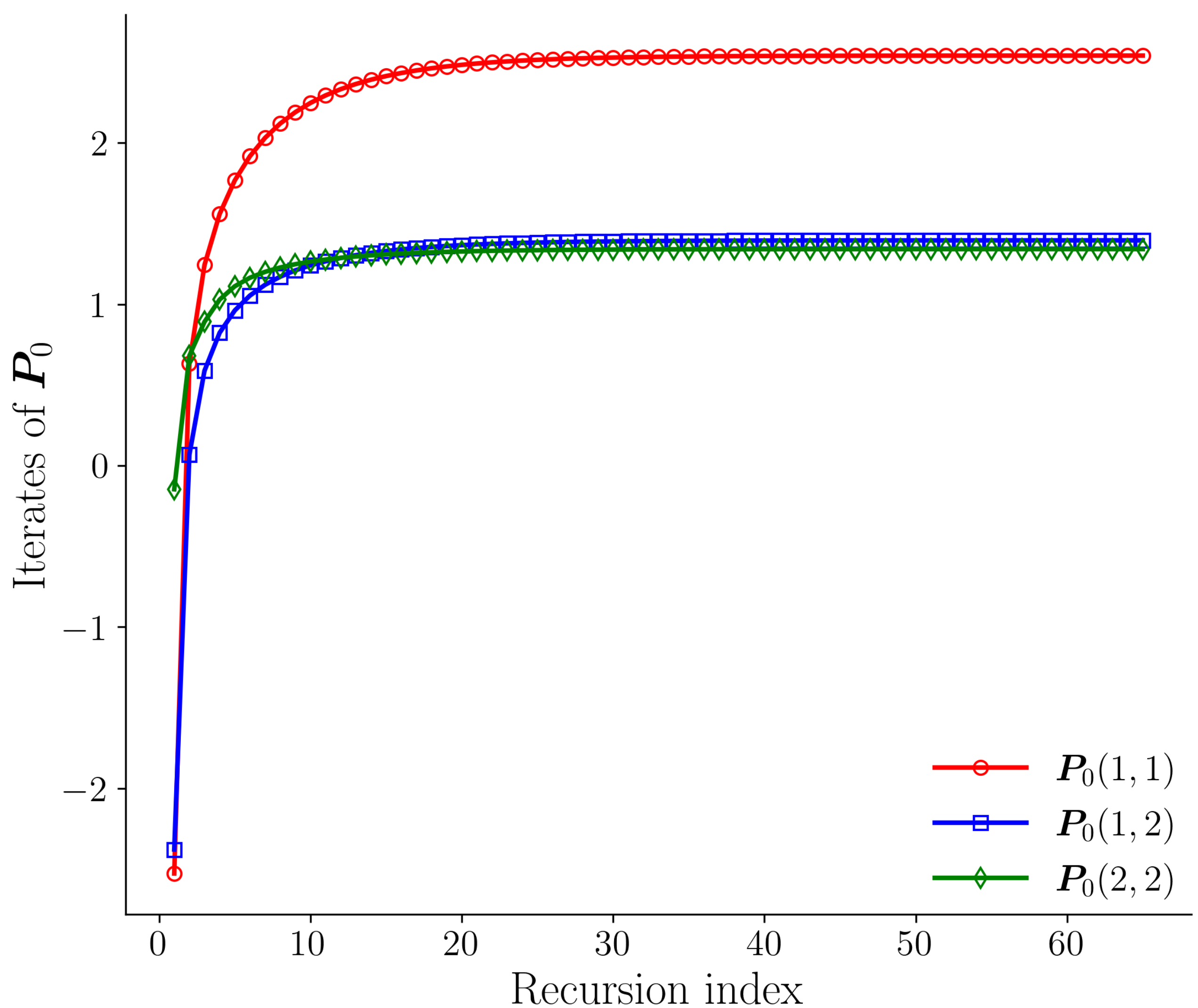
Controlled SDE:

$$\mathrm{d}x_{1t} = x_{2t}\,\mathrm{d}t,$$
$$\mathrm{d}x_{2t} = u_t\,\mathrm{d}t + \mathrm{d}w_t$$

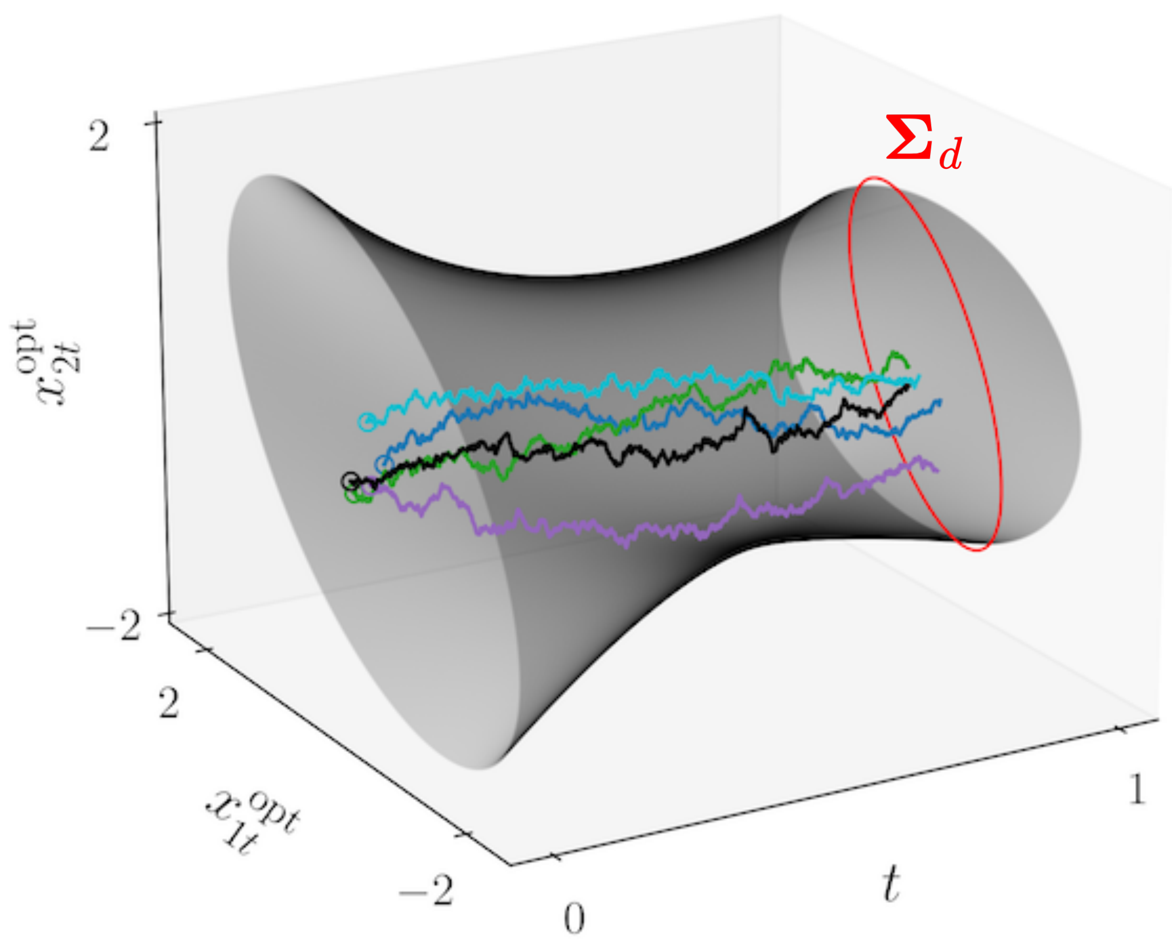
Parameters: $\boldsymbol{Q}_t \equiv \boldsymbol{I}$,

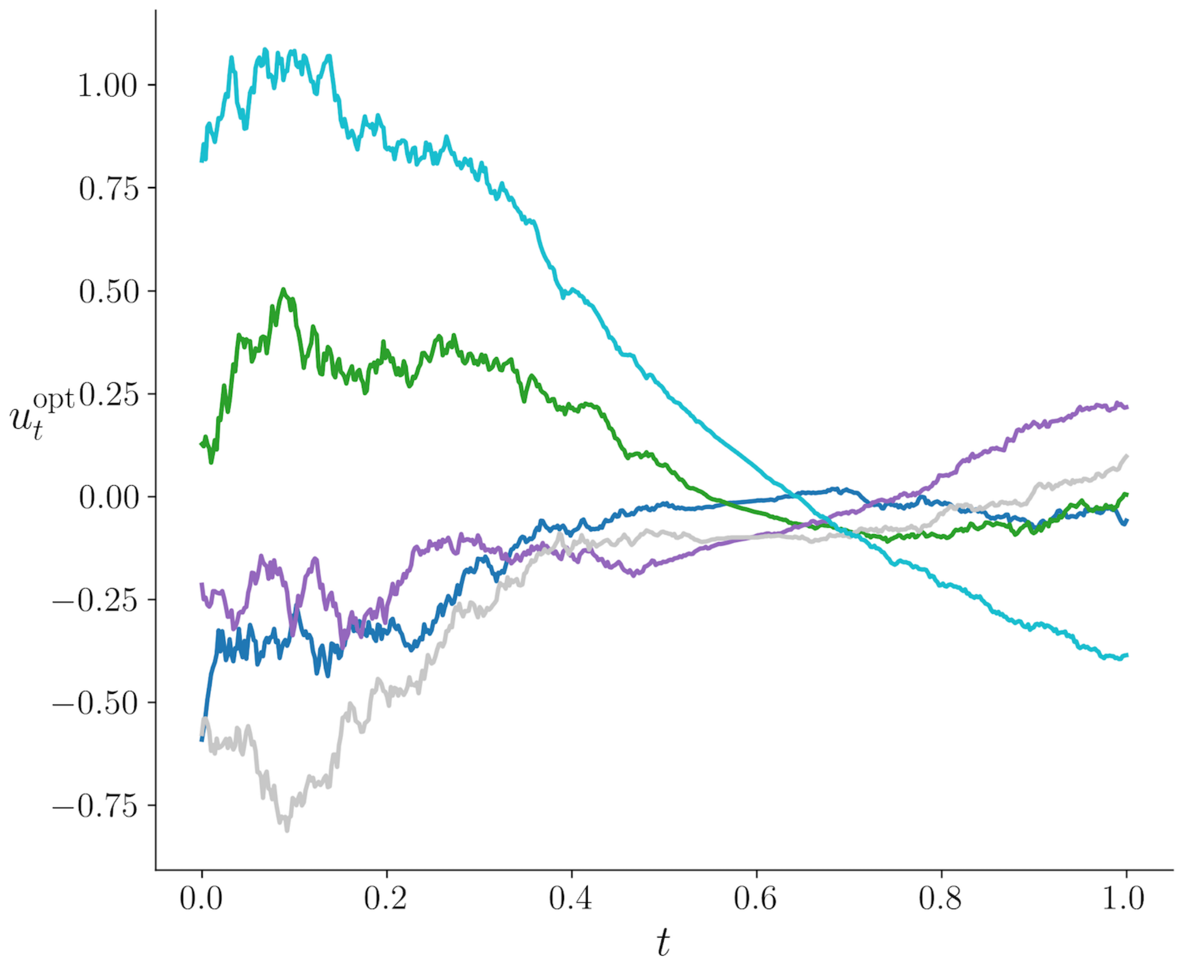
$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} 4.7295 & 1.9951 \\ 1.9951 & 3.6157 \end{bmatrix},$$

$$\boldsymbol{\Sigma}_d = \begin{bmatrix} 1.1189 & 0.7780 \\ 0.7780 & 1.7407 \end{bmatrix}$$

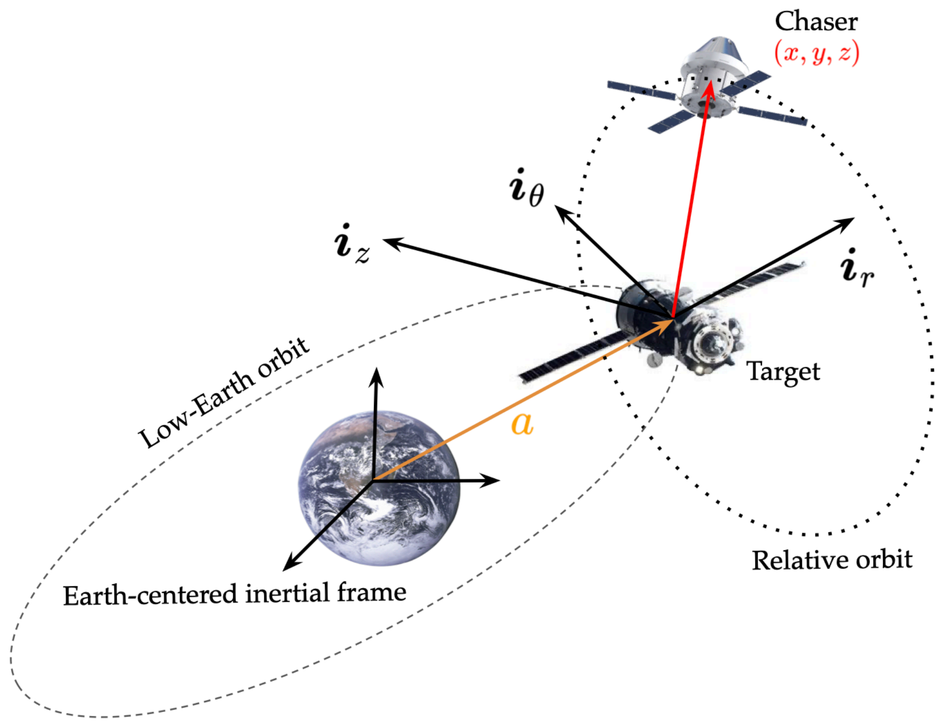


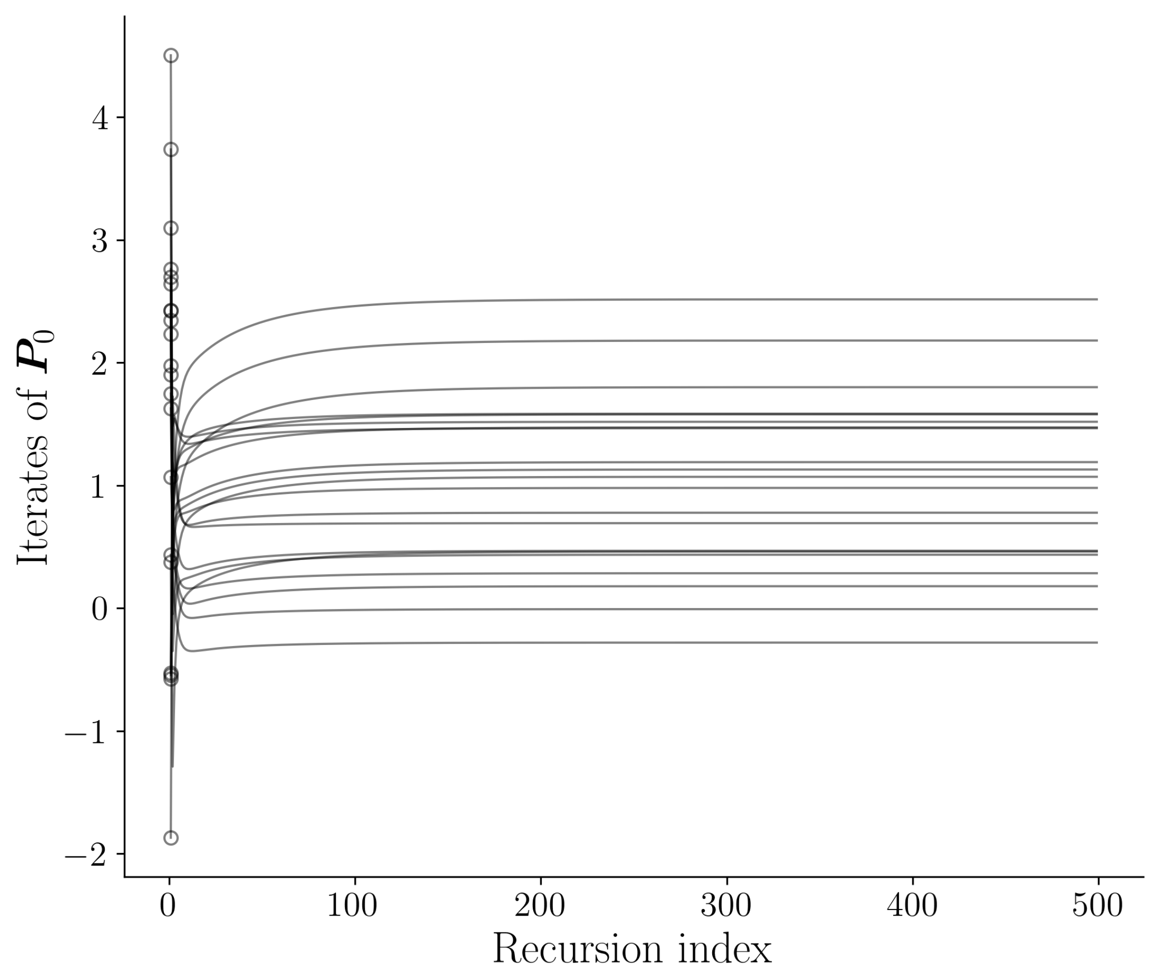
Numerical Case Study: Noisy Double Integrator





Numerical Case Study: Noisy Clohessy-Wiltshire





Controlled SDE:

$$\begin{aligned}
dx_{1t} &= x_{2t}\, dt,\\
dx_{2t} &= \left(3\nu^2 x_{1t} + 2\nu x_{4t} + u_{1t}\right) dt + dw_{1t},\\
dx_{3t} &= x_{4t}\, dt,\\
dx_{4t} &= \left(-2\nu x_{2t} + u_{2t}\right) dt + dw_{2t},\\
dx_{5t} &= x_{6t}\, dt,\\
dx_{6t} &= \left(-\nu^2 x_{5t} + u_{3t}\right) dt + dw_{3t},
\end{aligned}$$

Parameters:

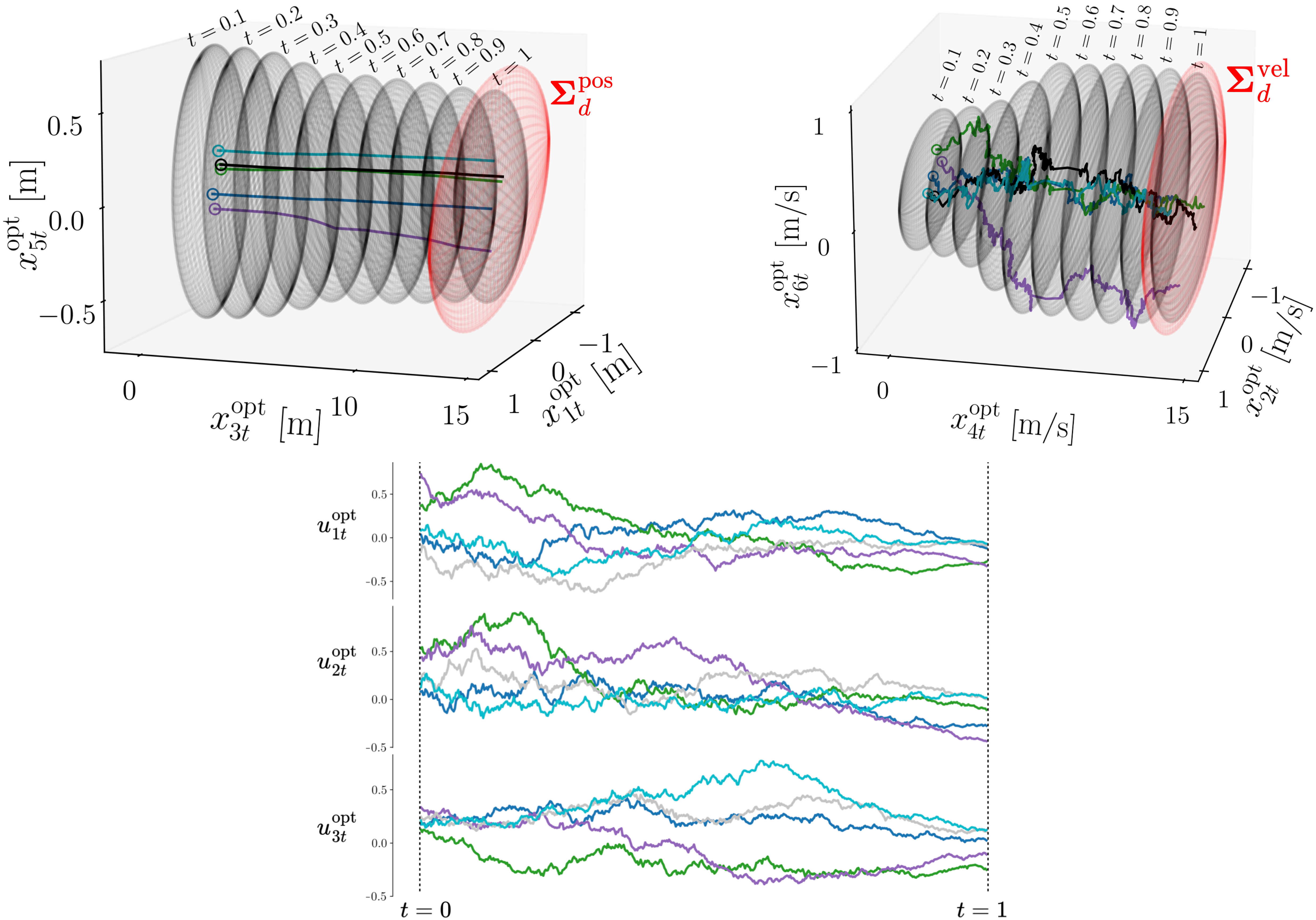
$$\nu := \sqrt{\mu/a^3} = 1.1276\times10^{-3}\ \text{rad/s}$$

$$\boldsymbol{Q}_t \equiv \boldsymbol{I},$$

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} 5.9148 & 3.8100 & 2.5815 & 2.1795 & 4.1628 & 1.9270\\ 3.8100 & 5.5664 & 2.8501 & 2.1819 & 3.8496 & 3.3638\\ 2.5815 & 2.8501 & 3.3834 & 1.5591 & 2.5389 & 2.3088\\ 2.1795 & 2.1819 & 1.5591 & 3.5850 & 2.6187 & 2.0098\\ 4.1628 & 3.8496 & 2.5389 & 2.6187 & 5.1285 & 2.5639\\ 1.9270 & 3.3638 & 2.3088 & 2.0098 & 2.5639 & 5.4354 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_d = \begin{bmatrix} 1.6431 & 1.1138 & 1.5453 & 1.1729 & 1.2916 & 0.4077\\ 1.1138 & 1.9581 & 1.4418 & 1.0926 & 1.2408 & 0.4495\\ 1.5453 & 1.4418 & 3.9142 & 1.9928 & 2.0221 & 1.5553\\ 1.1729 & 1.0926 & 1.9928 & 2.1027 & 1.3448 & 0.9645\\ 1.2916 & 1.2408 & 2.0221 & 1.3448 & 1.7077 & 0.7830\\ 0.4077 & 0.4495 & 1.5553 & 0.9645 & 0.7830 & 1.5008 \end{bmatrix}$$

Numerical Case Study: Noisy Clohessy-Wiltshire



Summary

Advances the state-of-the-art for solving LQ covariance control with terminal cost

Proposes fixed point recursion with convergence guarantee

Illustrates practical computation using two numerical case studies

Research products:

arXiv:2510:21944 [Under review in *IEEE TAC*] [To be presented in *AIAA Regional Conf, 2026*]

Generative Profiling for Soft Real-time Systems and its Applications to Resource Allocation [Accepted in *IEEE Real-Time and Embedded Technology and Applications Symposium (RTAS), 2026*]

Rasco: Resource Allocation and Scheduling Co-design for DAG Applications on Multicore [Published in *ACM Transactions on Embedded Computing Systems, 24(5s), pp. 1-27, 2025*]

Ongoing and Future Works

Optimal transport terminal cost:

$$\mathrm{OT}\left(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_d\right) = \mathrm{trace}\left(\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_d - 2\left(\boldsymbol{\Sigma}_1^{\frac{1}{2}}\boldsymbol{\Sigma}_d\boldsymbol{\Sigma}_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)$$

Bregman Divergence $D_\psi : \mathbb{S}_{++}^n \times \mathbb{S}_{++}^n \mapsto \mathbb{R}_{\geq 0}$ **terminal costs:**

Bregman divergence induced by mirror map $\psi : \mathbb{S}_{++}^n \mapsto \mathbb{R}$

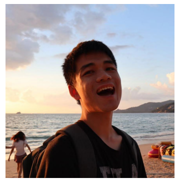
$$D_\psi(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_d) := \psi(\boldsymbol{\Sigma}_1) - \left\{\psi(\boldsymbol{\Sigma}_d) + \left\langle \frac{\partial \psi}{\partial \boldsymbol{\Sigma}_d}, \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_d \right\rangle\right\}$$

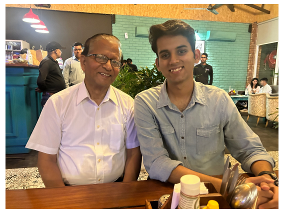
ψ	D_ψ	
$\lVert\boldsymbol{X}\rVert_{\mathrm{F}}^2$	$\lVert\boldsymbol{X} - \boldsymbol{Y}\rVert_{\mathrm{F}}^2$	This thesis
$\mathrm{trace}(\boldsymbol{X}\log\boldsymbol{X} - \boldsymbol{X})$	$\mathrm{trace}(\boldsymbol{X}\log\boldsymbol{X} - \boldsymbol{X}\log\boldsymbol{Y} - \boldsymbol{X} + \boldsymbol{Y})$	Quantum relative entropy
$-\log\det\boldsymbol{X}$	$\mathrm{trace}(\boldsymbol{X}\boldsymbol{Y}^{-1}) - \log\det(\boldsymbol{X}\boldsymbol{Y}^{-1}) - n$	Kullback-Leibler divergence

Acknowledgements

My heartfelt thanks to

↳ my advisor 

↳ my coauthors     

↳ my friends and family      

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Thank You