$\underline{d=2}$

$$\mathrm{vol}(\cdot) = \mu^2 t^2 \lim_{n\to\infty} \frac{1}{n^2} 2^2 \sum_{0\le i<j\le n} \left| \det(\xi_i \mid \xi_j) \right|$$

$$= 4\mu^2 t^2 \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n} \sum_{j=i+1}^{n} \det\begin{pmatrix} \frac{it}{n} & \frac{jt}{n} \\ 1 & 1 \end{pmatrix}$$

$$= 4\mu^2 t^3 \lim_{n\to\infty} \frac{1}{n^3} \underbrace{\sum_{i=0}^{n} \sum_{j=i+1}^{n} (j-i)}$$

Prove that: $\dfrac{n(n+1)(n+2)}{6}$

(next page)

Claim: $\displaystyle\sum_{i=0}^{n}\sum_{j=i+1}^{n}(j-i) = \frac{n(n+1)(n+2)}{6}$

Proof of claim:

$$\text{LHS} = \sum_{i=0}^{n}\left[\sum_{j=i+1}^{n} j - \sum_{j=i+1}^{n} i\right]$$

$$= \sum_{i=0}^{n}\left[\left(\sum_{j=1}^{n} j - \sum_{j=1}^{i} j\right) - \left(\sum_{j=1}^{n} i - \sum_{j=1}^{i} i\right)\right]$$

$$= \sum_{i=0}^{n}\left[\frac{n(n+1)}{2} - \frac{i(i+1)}{2} - ni + i^2\right]$$

$$= \frac{n(n+1)^2}{2} - \underline{\underline{\frac{1}{2}\sum_{i=0}^{n} i^2}} - \frac{1}{2}\sum_{i=0}^{n} i - n\sum_{i=0}^{n} i + \underline{\underline{\sum_{i=0}^{n} i^2}}$$

$$= \frac{n(n+1)^2}{2} + \frac{1}{2}\sum_{i=0}^{n} i^2 - \left(n+\frac{1}{2}\right)\frac{n(n+1)}{2}$$

$$\Rightarrow \text{LHS} = \frac{n(n+1)^2}{2} + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} - \left(n+\frac{1}{2}\right)\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\cancel{n}+1 + \frac{2n+1}{6} - \cancel{n} - \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{2}\underbrace{\left[\frac{3}{6} + \frac{2n}{6} + \frac{1}{6}\right]}$$

$$= \frac{n(n+1)}{2}\cdot\frac{2n+4}{6} = \frac{n(n+1)(n+2)}{6}$$

$$= \frac{(n^2+n)(n+2)}{6}$$

$$= \frac{n^3+3n^2+2n}{6} \quad \blacksquare$$

$$\therefore \lim_{n\to\infty} \frac{1}{n^3}\,\text{LHS} = \lim_{n\to\infty} \frac{1}{6}$$

$$\therefore 4\mu^2 t^3 \lim_{n\to\infty} \frac{1}{n^3}\,\text{LHS} = \frac{4}{6}\mu^2 t^3 = \frac{2}{3}\mu^2 t^3$$

$\underline{\text{for } d=3}$

$$\mathrm{Vol}(\cdot) = \mu^3 t^3 \lim_{n\to\infty} \frac{1}{n^3} \cdot 2^3 \cdot \sum_{0\le i<j<k\le n} \left|\det(\xi_i \mid \xi_j \mid \xi_k)\right|$$

$$= 8\mu^3 t^3 \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=0}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \begin{vmatrix} \frac{\left(\frac{it}{n}\right)^2}{2} & \frac{\left(\frac{jt}{n}\right)^2}{2} & \frac{\left(\frac{kt}{n}\right)^2}{2} \\ \frac{it}{n} & \frac{jt}{n} & \frac{kt}{n} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 8\mu^3 t^3 \lim_{n\to\infty} \frac{1}{n^3} \frac{\left(\frac{t}{n}\right)^2}{2} \cdot \left(\frac{t}{n}\right) \sum_{i=0}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \begin{vmatrix} i^2 & j^2 & k^2 \\ i & j & k \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 4\mu^3 t^6 \lim_{n\to\infty} \frac{1}{n^6} \sum_{i=0}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \underbrace{\begin{vmatrix} i^2 & j^2 & k^2 \\ i & j & k \\ 1 & 1 & 1 \end{vmatrix}}$$

$$= 4\mu^3 t^6 \lim_{n\to\infty} \frac{1}{n^6} \sum_{i=0}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} (k-j)(j-i)(k-i)$$

$$= 4\mu^3 t^6 \lim_{n\to\infty} \frac{1}{n^6} \cdot \frac{1}{180} n(n+1)^2 (n^3+4n^2+n-6)$$

$$= \frac{4\mu^3 t^6}{180} \underbrace{\lim_{n\to\infty} \frac{1}{n^6}(n^3+2n^2+n)(n^3+4n^2+n-6)}_{}$$

$$= \frac{\cancel{4}\mu^3 t^6}{\cancel{180}\ 45} \cdot 1$$

$$= \frac{\mu^3 t^6}{45} \qquad \blacksquare$$